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EXTENDING PARTIAL ISOMORPHISMS ON FINITE STRUCTURES

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We prove the following theorem:

Let A be a finite structure in a fixed finite relational language, $p_1,\dots,p_m$ partial isomorphisms of A. Then there exists a finite structure B, and automorphisms f_i of B extending the p_i's. This theorem can be used to prove the small index property for the random structure in this language. A special case of this theorem is, if A and B are hypergraphs. In addition we prove the theorem for the case of triangle free graphs.

Definition. Let L be a fixed finite relational language. Let $\mathcal{K}$ be a class of finite L-structures closed under substructures and isomorphisms and satisfying the Amalgamation Property (and the Joint Embedding Property): For all A, B and $\mathsf{C}\in\mathcal{K}$ (A possibly empty) with $\mathsf{A}\subset\mathsf{B}$ (A substructure of B) and $\mathsf{A}\subset\mathsf{C}$, there exists a finite L-structure $\mathsf{D}\in\mathcal{K}$, $\mathsf{A}\subset\mathsf{D}$, and embeddings $f:\mathsf{B}\to\mathsf{D}$ and $g:\mathsf{C}\to\mathsf{D}$ extending the identity on A. Up to isomorphism there is a unique countable structure M such that every finite substructure of M is in $\mathcal{K}$ and for all A, $\mathsf{B}\in\mathcal{K}$ with $\mathsf{A}\subset\mathsf{B}$ and $\mathsf{A}\subset M$, there exists an embedding of B into M extending the identity on A (see e.g. [1]). M is called the (countable) *universal homogeneous* $\mathcal{K}$*-structure*. If $\mathcal{K}$ is the class of all finite L-structures (resp. all finite graphs), M is also called the *random* L*-structure* (resp. random graph).

In [3] the small index property (SIP) is proved for ω-stable, ω-categorical countable structures and for the random graph. The key notion there is the notion of a generic sequence of automorphisms (of the structure under consideration). In the proof of the SIP for the random graph the following theorem of Hrushovski [2] is needed:

Let A be a finite graph, $p_1,\dots,p_m$ partial isomorphisms of A. Then there exists a finite graph B, and automorphisms $f_1,\dots,f_m$ of B extending $p_1,\dots,p_m$.

This theorem is used to prove that the set of "generic" (in the sense of [3]) n-tuples of automorphisms is comeager in G^n ($G=\mathrm{Aut}(M)$ with the usual topology), this again is used to prove SIP for the random graph M.

We extend Hrushovski's Theorem to an arbitrary finite relational language L answering a question in his paper [2], and also to the case of the triangle free graph. Our proofs use basically the same ideas as Hrushovski's proof, nevertheless the arguments in Theorem 2 are more intricate. The generalization to an arbitrary L also yields that one can find the domain of B and the f_i's only depending on the

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domain of A and the p_i's and not depending on the structure on A. The arguments in [3] and our theorems suffice to prove the small index property for the random relational L-structure (and for the homogeneous triangle free graph).

We first treat the (easier) case of the triangle free graph.

Definition. Let A be a graph. I.e. there is given a binary symmetric irreflexiv relation K on A. For $a \in \mathsf{A}$ denote by $N_{\mathsf{A}}(a)$ the set of all neighbours of a in A: $N_{\mathsf{A}}(a) := \{b \in \mathsf{A} \mid aKb\}$. A subset B of A is called *edge free* if there is no edge in A connecting two points in B. Note that the graph A is triangle free iff for all $a \in \mathsf{A}$ $N_{\mathsf{A}}(a)$ is edge free.

A *type*[1] over A is a set d of formulas of the form xKa or $\neg xKa$ $(a \in \mathsf{A})$ such that for any $a \in \mathsf{A}$: $xKa \notin d$ iff $\neg xKa \in d$. We say that a point b (of a supergraph $\mathsf{B} \supset \mathsf{A}$) *realizes* the type

$$d = \{xKa \mid a \in \mathsf{A},\ bKa\} \cup \{\neg xKa \mid a \in \mathsf{A},\ \text{not } bKa\}$$

over A; d is the type of the point b over A. In our case (the case of graphs) a type d is determined by a subset of A, namely the set $\{a \in \mathsf{A} \mid xKa \in d\}$, the set of neighbours of a point realizing d.

Notation. Let $p_1,\dots,p_m$ be the partial isomorphisms under consideration. Throughout the paper, D_i will denote the domain of p_i, R_i the range. We always suppose that for $1 \le i \le m$: $p_i^{-1} \in \{p_1,\dots,p_m\}$.

Lemma 1. *Let A be a finite triangle free graph. Let $p_1,\dots,p_m$ be partial isomorphisms on A. Then there exists a finite triangle free graph B, $\mathsf{A} \subset \mathsf{B}$ and automorphisms f_i on B extending p_i $(1 \le i \le m)$.*

Proof. The type realizing step:
A type of a point over A is determined by its set of neighbours in A, i.e. by an edge free subset B of A. For $j \in \omega$ let $F_j := \{E \mid E \subset \mathsf{A},\ E \text{ edge free},\ |E| = j\}$. Let $r := \max\{j \mid F_j \ne \emptyset\}$. For $E \in F_r$ define $s_E := |\{b \in \mathsf{A} \mid N_{\mathsf{A}}(b) = E\}|$ and $s_0 := \max\{s_E \mid E \in F_r\}$. Find a finite triangle free graph A_0, with $\mathsf{A} \subset \mathsf{A}_0$ such that for every $E \subset A$, $E \in F_r$: $|\{b \in \mathsf{A}_0 \mid N_{\mathsf{A}}(b) = E\}| = s_0$. This can be done by adding points "with type E over A" until the number s_0 is reached.

Now let A_n $(n < r)$ be already defined and numbers $s_0,\dots s_n$ be determined such that for $0 \le i \le n$ and $E \subset \mathsf{A}$, $E \in F_{r-i}$: $|\{b \in \mathsf{A}_n \mid N_{\mathsf{A}}(b) \supset E\}| = s_i$. Define for $E \in F_{r-(n+1)}$

$$s_E := |\{b \in \mathsf{A}_n \mid N_{\mathsf{A}}(b) \supset E\}|$$

and

$$s_{n+1} := \max\{s_E \mid E \in F_{r-(n+1)}\}.$$

Find a triangle free finite graph $\mathsf{A}_{n+1} \supset \mathsf{A}_n$ such that for $E \subset \mathsf{A}$, $E \in F_{r-(n+1)}$: $|\{b \in \mathsf{A}_{n+1} \mid N_{\mathsf{A}}(b) \supset E\}| = s_{n+1}$ without destroying this condition for $0 \le i \le n$. This can be done by adding for every $E \in F_{r-(n+1)}$ sufficiently many new points b with $N_{\mathsf{A}}(b) = E$.

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[1] This notion of type differs from the one used in mathematical logic. It would rather be called a $\{xKy\}$-type or quantifier free type without equality

Now let $\mathsf{C} := \mathsf{A}_r$. So for $E \in F_{r-i}$ $(1 \leq i \leq r)$ we have $|\{b \in \mathsf{C} \mid N_{\mathsf{A}}(b) \supset E\}| = s_i$. We want to prove that any partial isomorphism p of A, $p: D \to R$ has an extension to a bijection $h: \mathsf{C} \to \mathsf{C}$, such that whenever $a \in D$ and $c \in \mathsf{C}$ such that aKc, then $a^p K c^h$; i.e. the neighbours of a are mapped under h to the neighbours of a^p. We only have to show that that for $E_0 \subset D$, E_0 edge free (say $|E_0| = r - i$): $|\{s \in \mathsf{C} \mid N_{\mathsf{A}}(s) \cap D = E_0\}| = |\{s \in \mathsf{C} \mid N_{\mathsf{A}}(s) \cap R = E_0^p\}|$ (because then we can choose h restricted to the first set to be any bijection onto the second set extending p restricted to the first set. Note that every point which is in D and in the first set is mapped by p to a point which is in the second set. Here we need that p is a partial isomorphism).

We prove the equality by induction on i. Let $E_1, \dots, E_k$ be the sets E' with $E_0 \subsetneq E' \subset D$, E' edge free (possibly $k = 0$). These sets are mapped under p to the sets R' with $p(E_0) \subsetneq R' \subset R$, R' edge free. From the induction hypothesis and the choice of C follows:

$$|\{b \in \mathsf{C} \mid N_{\mathsf{A}}(b) \cap D = E_0\}| =$$
$$|\{b \in \mathsf{C} \mid N_{\mathsf{A}}(b) \supset E_0\}| - \sum_{i=1}^{k} |\{b \in \mathsf{C} \mid N_{\mathsf{A}}(b) \cap D = E_i\}| =$$
$$|\{b \in \mathsf{C} \mid N_{\mathsf{A}}(b) \supset E_0^p\}| - \sum_{i=1}^{k} |\{b \in \mathsf{C} \mid N_{\mathsf{A}}(b) \cap R = E_i^p\}| =$$
$$|\{b \in \mathsf{C} \mid N_{\mathsf{A}}(b) \cap R = E_0^p\}|.$$

The duplicator step:
Let $h_1, \dots, h_m$ be permutations on C extending $p_1, \dots p_m$, such that for $a \in D_i$ and $c \in \mathsf{C}$ with aKc: $a^{p_i} K c^{h_i}$. We can choose the h_i's such that $h_i^{-1} = h_j$ if $p_i^{-1} = p_j$. (In the case that $p_i^{-1} = p_i$ one has to choose the bijection h_i a little bit more careful, to ensure $h_i^{-1} = h_i$.) Let $\mathrm{Sym}(\mathsf{C})$ be the group of permutations on C and Γ be the subgroup generated by $h_1, \dots, h_m$. We define an equivalence relation $\equiv$ on $\mathsf{A} \times \Gamma$ by $(a^{p_{i_1} \cdots p_{i_s}}, \gamma) \equiv (a, h_{i_1} \dots h_{i_s} \gamma)$ for every $a \in A$, $\gamma \in \Gamma$ and $i_1, \dots, i_s$ such that $a^{p_{i_1} \cdots p_{i_s}}$ is defined. Note that for $a, b \in \mathsf{A}$ if $(a, \gamma_1) \equiv (b, \gamma_2)$ then $a^{\gamma_1} = b^{\gamma_2}$ and if $(a, 1) \equiv (b, \gamma)$ and bKb' then $b^\gamma = a$ and $b^\gamma K (b')^\gamma$ (here $(b')^\gamma$ need not be in A). For the last statement suppose $b^{p_{i_1} \cdots p_{i_k}} = a$ and $\gamma = h_{i_1} \dots h_{i_k}$. Now bKb' implies $b^{p_{i_1}} K (b')^{h_{i_1}}$ implies $\cdots$ implies $b^{p_{i_1} \cdots p_{i_k}} K (b')^{h_{i_1} \cdots h_{i_k}}$ that is $b^\gamma K (b')^\gamma$.

Let B be the graph with domain $A \times \Gamma/_{\equiv}$ defined by: aKb iff there is a $\gamma \in \Gamma$ and elements (a', γ), (b', γ) in the equivalence classes a and b such that $a'Kb'$. We embed A into B by identifying a with the equivalence class of $(a, 1)$. This is an injective mapping. Γ operates on B by $((a, \gamma')/_{\equiv})^\gamma = ((a, \gamma'\gamma)/_{\equiv})$ as a graph automorphism and h_i is an extension of p_i. We have to show that A is in fact a substructure of B under this embedding. For that we have to show that if a, a' is in A and $((a, 1)/_{\equiv}) K ((a', 1)/_{\equiv})$ is true in B then aKa' is true in A. So suppose $(a, 1) \equiv (b, \gamma)$ and $(a', 1) \equiv (b', \gamma)$ and bKb'. By what we have noted before we have $b^\gamma = a$, $(b')^\gamma = a'$ and $b^\gamma K (b')^\gamma$, that is aKa'.

Finally we have to check that B is triangle free. Suppose there is a triangle in B. That is there are $a, a', b, b', c, c' \in \mathsf{A}$ and $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$ such that aKa', $(a', \gamma_1) \equiv (b, \gamma_2)$, bKb', $(b', \gamma_2) \equiv (c, \gamma_3)$, cKc' and $(c', \gamma_3) \equiv (a, \gamma_1)$. We can suppose $\gamma_1 = 1$. By what we have noted before we have: $b^{\gamma_2}=a'$, $b^{\gamma_2}K(b')^{\gamma_2}$, $(b')^{\gamma_2}=c^{\gamma_3}$, $c^{\gamma_3}K(c')^{\gamma_3}$ and $(c')^{\gamma_3}=a$. So there is a triangle consisting of a, a' and $(b')^{\gamma_2}=c^{\gamma_3}$ in C. Contradiction. ∎

This completes the case of the triangle free graphs and we turn to the unrestricted relational case.

Theorem 2. *Let* A *be a finite relational structure. Let* $p_1,\dots,p_m$ *be partial isomorphisms of* A*. Then there exists a finite structure* B*, and automorphisms* $f_1,\dots,f_m$ *of* B *such that* A *is a substructure of* B *and for all* i $(1\le i\le m)$: f_i *is an extension of* p_i*.*

One idea of the proof is to look at a 3-ary relation R as being built up from binary relations $\{R_a\,|\,a\in\mathsf{A}\}$. An R-isomorphism –looked at in the new language– permutes these binary relations. This leads to the notion of a permorphism.

Definition 3. $\bar a$ will be an abbreviation for $a_1,\dots,a_n$. For L a relational language, L_j is the set of j-ary relation symbols in L. If χ is a permutation of L_j (for fixed j) and g a functions with domain E, we say g is a χ-*permorphism* if for every R in L_j, and every $a_0,\dots,a_n$ in E:

$$R(a_0,\dots,a_n)\iff R^{\chi}(a_0^g,\dots,a_n^g)$$

and if furthermore g is an isomorphism with respect to the other relation symbols in L.

We want to prove the theorem by induction on the maximal arity of relation symbols in the language. We need to prove the following generalization of the theorem:

Lemma 4. *Let* L *be a finite relational language and* A *a finite* L*-structure. Let* n *be the maximal arity of relation symbols in* L*. Let* $\chi_1,\dots\chi_m$ *be permutations of* L_n *and for* $1\le i\le m$ p_i *be a partial* χ_i*-permorphism on* A*, such that* $\chi_i^{-1}=\chi_j$ *if* $p_i^{-1}=p_j$*.*

There exists a finite L*-structure* B *and for* $1\le i\le m$ *a total* χ_i*-permorphism* f_i *of* B *extending* p_i*, such that* $f_i^{-1}=f_j$ *if* $p_i^{-1}=p_j$*.*

Proof. The proof is by induction on n. Let us first prove the induction step $n\to n+1$.

First step (the type realizing step):
We construct a finite extension C_1 of A and bijections φ_i on C_1 extending p_i (for $1\le i\le m$) such that $\varphi_i^{-1}=\varphi_j$ if $p_i^{-1}=p_j$ and for $a\in\mathsf{C}_1$, $\bar a\in D_i^n$ and $R\in\mathsf{L}_{n+1}$:

$$R(a,\bar a)\iff R^{\chi_i}(a^{\varphi_i},\bar a^{p_i}).$$

Let Δ be the set of all formulas $R(x,y_1,\dots,y_n)$ $(R\in\mathsf{L}_{n+1})$. A Δ-type over a set G is a set d of formulas of the form $R(x,\bar a)$ or $\neg R(x,\bar a)$ $(\bar a\in G^n,\ R\in\mathsf{L}_{n+1})$ such that for any $\bar a\in G$: $R(x,\bar a)\notin d$ iff $(\neg R(x,\bar a))\in d$. In other words: A Δ-type over G decides for every $\bar a\in G^n$ and every $R\in L_{n+1}$ if $R(x,\bar a)$ holds or not, i.e. a Δ-type over G is determined by a function from G^n to the power set of L_{n+1}.

Let G be a subset of a structure A, $a\in\mathsf{A}$, and d a Δ-type over G. We say that a realizes d if for all $\bar a\in G^n$: $(R(a,\bar a)$ iff $R(x,\bar a)\in d)$; we define $\Delta\text{-tp}(a/G)$ to be the unique Δ-type over G which a realizes. Without loss of generality we suppose that every $a \in \mathsf{A}$ realizes a different Δ-type over A (it is easy to extend A to get this property).

Now let C_1 be the set of all Δ-types over A. We embed A into C_1 via the function *: $a^* := \Delta\text{-tp}(a/A)$ (for $a \in A$). For $R \in \mathsf{L}_{n+1}$ we define $R(d, d_1, \ldots, d_n)$ iff there is $\bar{a} \in \mathsf{A}$ such that $d_1 = a_1^*, \ldots, d_n = a_n^*$ and $R(x, \bar{a}) \in d$ (so $\Delta\text{-tp}(d/A) = d$). Now A is a substructure of C_1 and every Δ-type over A is realized by exactly one element of C_1. Let $1 \leq i \leq m$.

For a Δ-type s over D_i we will consider the set of realizations $s(\mathsf{C}_1) := \{b \in \mathsf{C}_1 \mid b$ realizes s over $D_i\}$. Furthermore we transport such a Δ-type s by p_i: We define $\hat{p_i}(s)$ to be the Δ-type over R_i satisfying

$$R^{\chi_i}(x, \bar{a}^{p_i}) \in \hat{p_i}(s) \iff R(x, \bar{a}) \in s$$

(for $\bar{a} \in D_i$).

For all Δ-types s over D_i the cardinality of $s(\mathsf{C}_1)$ is the same and equals furthermore the cardinality of $\hat{p_i}(s)(\mathsf{C}_1)$ (namely equals the number of functions from $A^n - D_i^n$ to the power set of L_{n+1}).

Let $\varphi_i \in \mathrm{Sym}(\mathsf{C}_1)$ be an extension of p_i such that for every Δ-type s over D_i: φ_i maps $s(\mathsf{C}_1)$ to $\hat{p_i}(s)(\mathsf{C}_1)$. Note: Because p_i is a χ_i-permorphism every element of $D_i \cap s(\mathsf{C}_1)$ is mapped by p_i to an element of $R_i \cap \hat{p_i}(s)(\mathsf{C}_1)$. Now we have indeed that for $a \in C_1$, $\bar{a} \in D_i^n$ and $R \in \mathsf{L}_{n+1}$: $R(a, \bar{a}) \iff R^{\chi_i}(a^{\varphi_i}, \bar{a}^{p_i})$. Furthermore, it is easy to find the φ_i in such a way that $\varphi_i^{-1} = \varphi_j$ if $p_i^{-1} = p_j$.

Second step (the permorphism step):
Now we consider the structure A in a new language:

For every $R \in \mathsf{L}_{n+1}$ and every $a \in \mathsf{C}_1$ we introduce a new symbol R_a.

$\mathsf{L}^* := \mathsf{L} - \mathsf{L}_{n+1} \cup \{R_a \mid R \in \mathsf{L}_{n+1},\ a \in \mathsf{C}_1\}$ and we define the L^*-structure A^* with domain A by $R_a(\bar{b}) :\iff R(a, \bar{b})$ (for $\bar{b} \in A$) for the new symbols. We define permutations ψ_i on L_n^* by $(R_a)^{\psi_i} := R^{\chi_i}_{a^{\varphi_i}}$ and for the other symbols S by $S^{\psi_i} := S$. p_i is a partial ψ_i-permorphism (for $1 \leq i \leq m$). By induction we get a finite extension C_2^* of A and total ψ_i-permorphisms h_i extending p_i $(1 \leq i \leq m)$, such that $h_i^{-1} = h_j$ if $p_i^{-1} = p_j$.

Third step (the duplicator step):
Consider the finite group $\mathrm{Sym}(\mathsf{C}_2^*) \times \mathrm{Sym}(\mathsf{C}_1) \times \mathrm{Sym}(\mathsf{L}_{n+1}))$. Let Γ be the subgroup generated by $\delta_i := (h_i, \varphi_i, \chi_i)$ $(1 \leq i \leq m)$. We define a L-structure on $\mathsf{A} \times \Gamma$: For $R \in \mathsf{L}_{n+1}$:

$$R((a_0, \gamma_0), \ldots, (a_n, \gamma_n)) \quad \text{iff} \quad \gamma_0 = \ldots = \gamma_n =: (h, \varphi, \chi)$$

and $R^{\chi^{-1}}(a_0, \ldots, a_n)$. For every other symbol S in L:

$$S((a_0, \gamma_0), \ldots, (a_l, \gamma_l)) \quad \text{iff} \quad \gamma_0 = \ldots = \gamma_l \quad \text{and} \quad S(a_0, \ldots, a_l).$$

We define an equivalence relation $\equiv$ on $\mathsf{A} \times \Gamma$ by $(a^{p_{i_1} \cdots p_{i_s}}, \gamma) \equiv (a, \delta_{i_1} \ldots \delta_{i_s} \gamma)$ for every $a \in A$, $\gamma \in \Gamma$ and $i_1, \ldots, i_s$ such that $a^{p_{i_1} \cdots p_{i_s}}$ is defined. Later on we will need the following fact: If $(a, 1) \equiv (b, (h, \varphi, \chi))$, then $a = b^h = b^{\varphi}$.

Let B be the structure with domain $A \times \Gamma/_{\equiv}$ defined by: $S(b_0, \ldots, b_s)$ iff there are elements a_i in the equivalence classes b_i such that $S(\bar{a})$ (in $\mathsf{A} \times \Gamma$) (for every S in L).

We embed A into B by identifying a with $(a,1)/_{\equiv}$. Γ operates canonically on B and δ_i is an extension of p_i. δ_i is a χ_i-permorphism. It remains to show that A is in fact a substructure of B. I.e. if $R \in \mathsf{L}_{n+1}$, $a, \bar{a} \in$ A and $R((a,1)/_{\equiv},(a_1,1)/_{\equiv},\ldots,(a_n,1)/_{\equiv})$ is true in B, then $R(a,\bar{a})$ is true in A (the case of the other symbols in L is easier). So let $\gamma=(h,\varphi,\chi)$ be in Γ and $b,\bar{b}$ be in A, such that $(a,1)\equiv(b,\gamma)$, $(a_1,1)\equiv(b_1,\gamma)$, $\ldots$,$(a_n,1)\equiv(b_n,\gamma)$ and $R^{\chi^{-1}}(b,\bar{b})$. We define the permutation ψ on L_n^* by $(R_a)^{\psi}=R^{\chi}_{a^{\varphi}}$ ($S^{\psi}=S$ for the other symbols).

Now we will need the fact, that h is a ψ-permorphism of C_2^*. That follows for generators of Γ (i.e. there is an i such that $h=h_i$; $\varphi=\varphi_i$; $\chi=\chi_i$) from the second step, and extends to the whole group Γ.

We have $b^{\varphi}=a$ and $\bar{b}^h=\bar{a}$. From $(R^{\chi^{-1}})_b(\bar{b})$ follows $(R^{\chi^{-1}})^{\chi}_{b^{\varphi}}(\bar{b}^h)$, that is $R_a(\bar{a})$.

Case $n=1$:

The construction in this case is similar to the first step.

Let Δ be $\{R(x)\,|\,R\in\mathsf{L}_1\}$. In this case a Δ-type contains formulas of the form $R(x)$ or $\neg R(x)$. For a Δ-type d, let n_d be the number of realizations of d in A, and let k be the maximum of these numbers. Let B be the disjoint union of k copies of the set of all Δ-types. One can embed A into B and extend p_i to a χ_i-permorphism f_i of B (for $1\leq i\leq m$), such that $f_i^{-1}=f_j$ if $p_i^{-1}=p_j$. ∎

Let us explain how the case of A being a hypergraph is covered by Theorem 2. We consider the hypergraph A as being built up by its r-uniform parts ($1\leq r\leq |\mathsf{A}|$). For every such r-uniform part we put a r-ary relation symbol into L. This way the hypergraph A is translated to a L-structure A′ and the theorem yields a L-structure B′, which can be translated back to a hypergraph B. For that one has to ensure that B′ is a symmetric L-structure (as A′ is). But the proof of Theorem 2 yields automatically a symmetric L-structure B if one starts with a symmetric A. (This also follows from the theorem not only from the proof: If one starts with A being symmetric and gets a possibly non symmetric B, then one can erase every non symmetric instance of any relation to get a symmetric structure B^s, which does the job.)

Let $G=\mathrm{Aut}(M)$ be the automorphism group of a countable structure M. G can be considered as a topological group, where $\{G_A\,|\,A\subset M, A \text{ finite}\}$ forms a basis for the neighbourhood of the identity; $G_A=\{g\in G\,|\,\forall a\in A:\ a^g=a\}$. The index of every open subgroup is at most countable. The SIP states the converse: We say the small index property holds for M, if every subgroup U of G with $[G:U]<2^{\omega}$ is open. See [3] for a survey which structures are proved to have the SIP.

Theorem 5. a) *Let* L *be a fixed finite relational language. Then the small index property holds for the countable random* L*-structure.*

b) *The small index property holds for the universal homogeneous triangle free graph.*

Proof. This follows from Theorem 2 respectively from Lemma 1 in the same way as the small index property for the random graph in [3] follows from Hrushovski's Theorem. ∎

Before Hrushovski proved his extension theorem in the case of graphs there had already been a proof of Truss for the case of only one partial isomorphism on a graph (see [4], [2]). This proof also works in the case of an arbitrary L-structure (but only for one isomorphism). In fact the proof shows that in this case one can find the super structure B in a certain sense (see the following corollary) independently of the structure on A. On the other hand Hrushovski's extension graph B depends on the structure on A. From Theorem 2 it follows that one can choose B independently from the structure on A even in the case of finitely many partial isomorphisms.

Corollary 6. *Let A be a finite set and $p_1, \dots, p_m$ be partial injective mappings on A. Then there exists a finite set B and bijections $f_1, \dots, f_m$ on B extending the p_i's with the property that for every relational structure* A *with domain A such that $p_1, \dots, p_m$ are partial isomorphisms of* A *there is a structure* B *with domain B such that* A *is substructure of* B *and the f_i's are automorphisms of* B. *("B and the f_i's can be chosen independently from the structure on* A.")

Proof. Equip A with every relation R of arity $\leq |A|$ such that all the p_i's are R-isomorphisms to get the structure A and the language L. Theorem 2 yields a finite L-structure B and total L-automorphisms $f_1, \dots, f_m$ extending the p_i's. The domain B and the f_i's satisfy the requirements of the corollary. ∎

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